§3.1 Introduction to Determinants

Recall from §2.2, if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ a 2×2 matrix then A is invertible if and only if $ad-bc \neq 0$.

This is a nice criteria to check for invertibility so we'd like something similar for $n\times n$ matrices with $n > 2$.

<u>Defn</u>: With $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ as before, the <u>determinant</u> of A is $\underline{\det A} = ad-bc$.

Now this (right now) only works for 2×2 matrices so let's adapt it to 3×3 matrices.

Let A be the 3×3 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

here a_{ij} is the entry in the $\begin{cases} i^{th} \text{ row} \\ j^{th} \text{ column} \end{cases}$

Now let A_{ij} denote the 2×2 obtained by deleting row i and row j.

Example : With A as before

$$A_{12} = \begin{bmatrix} \cancel{a_{11}} & \cancel{a_{12}} & \cancel{a_{13}} \\ a_{21} & \cancel{a_{22}} & a_{23} \\ a_{31} & \cancel{a_{32}} & a_{33} \end{bmatrix}$$

$$= \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix}$$

Now this a 2×2 matrix and we know its determinant.

<u>Defn</u> : With A the 3×3 matrix as before

$$\underline{\det A} = a_{11} \det(A_{11}) - a_{12} \det(A_{12}) + a_{13} \det(A_{13})$$

<u>Example</u>: Compute $\det A$ where $A = \begin{bmatrix} 1 & 4 & 3 \\ 2 & 0 & 1 \\ 6 & 2 & 1 \end{bmatrix}$

$$\det A = 1 \cdot \det \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} - 4 \det \begin{bmatrix} 2 & 1 \\ 6 & 1 \end{bmatrix} + 3 \det \begin{bmatrix} 2 & 0 \\ 6 & 2 \end{bmatrix}$$

$$= 1 \cdot (0 - 2) - 4 (2 - 6) + 3 (4 - 0)$$

$$= -2 + 16 + 12$$

$$= \boxed{26}$$

- With this we can generalize the definition to any $n \times n$ matrix for $n \geq 2$.

Defn: Let $A = (a_{ij})$ be an $n \times n$ matrix for $n \geq 2$

$$\det A = a_{11} \det A_{11} - a_{12} \det A_{12} + \cdots + (-1)^{1+n} a_{1n} \det A_{1n}$$
$$= \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det A_{1j}$$

Remarks

- Notice each A_{1j} is an $n-1 \times n-1$ matrix so the definition of the determinant of an $n \times n$ matrix is recursively defined in terms of determinants of $n-1 \times n-1$ matrices.

- Notice the coefficients $a_{11}, a_{12}, \ldots, a_{1n}$ above come from the first row of A.

  - Could we somehow choose a different row?
  - Or maybe a column?

Answer: Yes! But we need some new notation first.

Defn: Let $A = (a_{ij})$ be an $n \times n$ matrix. The (i,j)-cofactor of A is

$$C_{ij} = (-1)^{i+j} \det A_{ij}$$

This is nice in the sense that it keeps track of the alternating signs. With this we can rewrite the determinant from earlier as

$$\det A = a_{11} C_{11} + a_{12} C_{12} + \cdots + a_{1n} C_{1n} = \sum_{j=1}^{n} a_{1j} C_{1j}$$

and this is referred to as the cofactor expansion along the first row of A.

Theorem: The determinant can be compute via a cofactor expansion along any row or column.

Expansion across k^{th} row is

$$\det A = a_{k1} C_{k1} + a_{k2} C_{k2} + \cdots + a_{kn} C_{kn} = \sum_{j=1}^{n} a_{kj} C_{kj}$$

Expansion across ℓ^{th} column is

$$\det A = a_{1\ell} C_{1\ell} + a_{2\ell} C_{2\ell} + \cdots + a_{n\ell} C_{n\ell} = \sum_{j=1}^{n} a_{j\ell} C_{j\ell}$$

Example: We showed $\det \begin{bmatrix} 1 & 4 & 3 \\ 2 & 0 & 1 \\ 6 & 2 & 1 \end{bmatrix} = 26$

by expanding along row 1.

Lets double check and expand across row 2:

$$\det A = 2 \cdot -\det \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} + \cancel{0 \cdot \det \begin{bmatrix} 1 & 3 \\ 6 & 1 \end{bmatrix}} + 1 \cdot -\det \begin{bmatrix} 1 & 4 \\ 6 & 2 \end{bmatrix}$$

multiplying by zero!

$$= -2(4-6) - 1(2-24)$$

$$= 4 + 22$$

$$= \boxed{26}$$

What about column 2? Let's expand across it.

$$\det A = 4 \cdot -\det \begin{bmatrix} 2 & 1 \\ 6 & 1 \end{bmatrix} + 0 \cdot \det \begin{bmatrix} 1 & 3 \\ 6 & 1 \end{bmatrix} + 2 \cdot -\det \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$$

$$= -4(2-6) - 2(1-6)$$

$$= 16 + 10$$

$$= \boxed{26}$$

Remarks

- This is useful since we can choose to expand across rows or columns with zeroes.

- Be careful with the signs! Notice the $\pm$ on C_{ij} depends only on the position of a_{ij} in A. ~~##~~ With this, the $(-1)^{i+j}$ gives a "checkerboard pattern" of signs

$$\begin{bmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Example: Compute $\det A$ where

$$A = \begin{bmatrix} 5 & 0 & -7 & 4 & -5 \\ 0 & 0 & 1 & 0 & 0 \\ 7 & 2 & -6 & 4 & -7 \\ 4 & 0 & 6 & 2 & -4 \\ 0 & 0 & 8 & -1 & 3 \end{bmatrix}$$

Let's pick "good" rows and columns to save time!

Lets expand along row 2 since it has lots of zeroes!

$$\det A = \cancel{0 \cdot C_{21}} + \cancel{0 \cdot C_{22}} + 1 \cdot -\det\begin{bmatrix} 5 & 0 & 4 & -5 \\ 7 & 2 & 4 & -7 \\ 4 & 0 & 2 & -4 \\ 0 & 0 & -1 & 3 \end{bmatrix} + \cancel{0 \cdot C_{24}} + \cancel{0 \cdot C_{25}}$$

$$= -\det\begin{bmatrix} 5 & 0 & 4 & -5 \\ 7 & 2 & 4 & -7 \\ 4 & 0 & 2 & -4 \\ 0 & 0 & -1 & 3 \end{bmatrix}$$

column 2 has lots of zeroes!

$$= -\left(\cancel{0 \cdot C_{12}} + 2\det\begin{bmatrix} 5 & 4 & -5 \\ 4 & 2 & -4 \\ 0 & -1 & 3 \end{bmatrix} + \cancel{0 \cdot C_{32}} + \cancel{0 \cdot C_{42}}\right)$$

$$= -2\det\begin{bmatrix} 5 & 4 & -5 \\ 4 & 2 & -4 \\ 0 & -1 & 3 \end{bmatrix}$$

column 1 has a zero!

$$= -2\left(5\det\begin{bmatrix} 2 & -4 \\ -1 & 3 \end{bmatrix} - 4\det\begin{bmatrix} 4 & -5 \\ -1 & 3 \end{bmatrix} + \cancel{0 \cdot C_{31}}\right)$$

$$= -2(5(6-4) - 4(12-5))$$

$$= -2(10 - 28)$$

$$= \boxed{36}$$

Defn

a) An $n \times n$ matrix A is <u>upper-triangular</u> if all entries below the main diagonal are zero.

$$A = \begin{bmatrix} a_{11} & \cdots & & a_{1n} \\ & a_{22} & & \vdots \\ & & \ddots & \vdots \\ \bigcirc & & & a_{nn} \end{bmatrix}$$

b) A $n \times n$ matrix A is <u>lower-triangular</u> if all entries ~~below~~ above the main diagonal are zero

$$A = \begin{bmatrix} a_{11} & & & \bigcirc \\ \vdots & a_{22} & & \\ \vdots & & \ddots & \\ a_{n1} & \cdots & \cdots & a_{nn} \end{bmatrix}$$

Theorem

If A is triangular (upper or lower), $\det A$ is the Product of the entries on the main diagonal

$$\det A = a_{11} a_{22} \cdots a_{nn}$$

Proof: Think about it. What rows/columns would be good to expand along?